INF 556: Topological Data Analysis **Fall 2018**

Lecture 4: Introduction to homology (part 2)

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Disclaimer:
Some typo and errors may remain. Please mention them at `theo.lacombe@polytechnique.edu`. Use these notes with caution, especially during the exam (we decline all responsibility linked with the use of these notes during the exam session).

Reminder: These notes are a concise summary of the lectures. They do not intend in any case to substitute to your personal notes and are just an additional support in order to clarify or insist on some points.

Some references: As a complement for these short lecture notes, you can check for two books:

- *Element of Algebraic Topology*, by J.Munkres (1984), especially chapter 1.
- *Introduction to Computational Topology*, by H.Edelsbrunner and J.Harer (you can find an extract relative to Homology theory on the website for this course).

Keywords: Simplicial homology, homotopy equivalence invariant, singular homology (optional).

4.1 Algorithm to compute homology

Input: A finite simplicial complex K, a field $\mathbb{K}$.

Output: $H_r(K, \mathbb{K})$, $\forall r \geqslant 0$.
Remind that $H_r(K, \mathbb{K}) \simeq \mathbb{K}^n$ for some $n \in \mathbb{N}$, so we basically just need to find $\dim(H_r(K, \mathbb{K})) = \underbrace{\dim(Z_r)}_{\dim(\ker(\partial_r))} - \underbrace{\dim(B_r)}_{\mathrm{rk}(\partial_{r+1})}$ (a.k.a β_r).

Therefore, the algorithm consists on computing M_r the matrix of ∂_r in the simplex basis (for all r). A fundamental result of linear algebra (rank-nullity theorem[1]) gives the following result:

$$\begin{array}{cc} & \begin{array}{ccc} \sigma_1 & \cdots & \sigma_{|K_r|} \end{array} \\ \begin{array}{c} \nu_1 \\ \\ \vdots \\ \\ \nu_{|K_{r-1}|} \end{array} & \begin{bmatrix} & & \\ & & \\ & M_r & \\ & & \\ & & \end{bmatrix} \end{array} \longrightarrow dim(H_r) = |K_r| - rk(M_r) - rk(M_{r+1})$$

In practice: There are different approach. The easiest-to-implement one is the *Gaussian Elimination*. Despite being computationally costly in theory ($\mathcal{O}(n^3)$), it appears to be efficient in practice because the matrix of ∂_r is generally sparse (near-linear time in practice).[2]

[1] *theoreme du rang* in French)
[2] Make use of it in your implementations!

$$\begin{bmatrix} 1 & 1 & 0 \\ \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & \boxed{1} \end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_2} \begin{bmatrix} 1 & 1 & 1 \\ \boxed{1} & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 \end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_1} \begin{bmatrix} 1 & 1 & 0 \\ \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix}$$

Figure 4.1: Sketch of Gaussian elimination

For the following algorithm, we define:

$$\text{low}(j) = \begin{cases} 0 & \text{if } M[i,j] = 0, \forall i \\ \max\{i | M[i,j] \neq 0\} & \text{otherwise} \end{cases}$$

which is the row of the lowest non zero-entry in column j.

Algorithm 1 Calculate the rank of a matrix with Gaussian elimination

for $j = 1..|K_r|$ **do**
  while $\exists i < j$ s.t. $\text{low}(i) = \text{low}(j) \neq 0$ **do**
    $c_j \leftarrow c_j - \frac{M[j,\text{low}(i)]}{M[i,\text{low}(i)]} c_i$
  end while
end for
return $|\{j | \text{low}(j) \neq 0\}|$

Proposition 1. *This algorithm converges and is correct.*

Proof. Exercise. □

4.2 Morphisms

The field $\mathbb{K}$ is fixed.

We have an operator $H_r : K \mapsto H_r(K, \mathbb{K})$. We want to extend its definition to maps as well: $(f : K \to L) \mapsto H_r(f)$. We also want to do it functorially, which means that we want to have:

$$H_r(f \circ g) = H_r(f) \circ H_r(g)$$
$$H_r(id_K) = id_{H_r(K)}$$

Definition 1. $f : K \to L$ *is* simplicial *if* $\exists f_0 : K_0 \to L_0$ *such that* $\forall \sigma = \{v_0..v_n\} \in K, f(\sigma) = \{f_0(v_0) \dots f_0(v_n)\}$.

Remark: In this situation, f is entirely defined by its restriction to the set of vertices, i.e. by f_0. In the following, both functions are identified.

Proposition 2. *A simplicial map $f : K \to L$ induces a chain map $f_\# : C_r(K) \to C_r(L)$ between the chain spaces of K and L which verifies $f_\# \circ \partial_k = \partial_k \circ f_\#$.*

$$\begin{tikzcd} \cdots \arrow[r] & C_r(K) \arrow[r, "\partial_r"] \arrow[d, "f_\#"] & C_{r-1}(K) \arrow[r, "\partial_{r-1}"] \arrow[d, "f_\#"] & \cdots \arrow[r, "\partial_1"] & C_0(K) \arrow[r, "\partial_0"] \arrow[d, "f_\#"] & 0 \\ \cdots \arrow[r] & C_r(L) \arrow[r, "\partial_r"] & C_{r-1}(L) \arrow[r, "\partial_{r-1}"] & \cdots \arrow[r, "\partial_1"] & C_0(L) \arrow[r, "\partial_0"] & 0 \end{tikzcd}$$

Figure 4.2: Chain map. $f_\#$ transforms chains in K into chains in L, while respecting the structure.

Proof. Define $f_\#$ on each simplex σ as follows:

$$f_\#(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = \dim(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

and then extend it by linearity:

$$f_\# \underbrace{\left(\sum_i \alpha_i \sigma_i \right)}_{\in C_r(K)} := \sum_i \alpha_i \underbrace{f_\#(\sigma_i)}_{\in C_r(L)}$$

Then we observe that $\forall \sigma = [v_0 \dots v_r] \in C_r(K)$:

$$\begin{aligned} f_\# \circ \partial_r(\sigma) &= f_\# \left(\sum_{i=0}^{r} (-1)^i [v_0 \dots \widehat{v_i} \dots v_r] \right) \\ &= \sum_{i=0}^{r} (-1)^i \qquad \underbrace{f_\# \left([v_0 \dots, \widehat{v_i}, \dots, v_r] \right)}_{= \begin{cases} [f(v_0) \dots, \widehat{f(v_i)}, \dots, f(v_r)] & \text{if } \dim = r-1 \\ 0 & \text{otherwise} \end{cases}} \end{aligned}$$

But if $\dim = r - 1$ for some i, then $\dim f(\sigma) < r$ as well.

We can then check $f_\# \circ \partial_r(\sigma) = \partial_r \circ f_\#(\sigma)$ for all σ. We do not provide the details here, you can do the computations by yourself. You will need to consider the three cases: $\dim f(\sigma) = r$, $\dim f(\sigma) = r - 1$, $\dim f(\sigma) \leqslant r - 2$. $\square$

Corollary 1. *A simplicial map $f : K \to L$ induces a linear map $f_* : H_r(K) \to H_r(L), \forall r$.*

Proof. The core argument is that the chain map $f_\#$ goes to the quotient. Since everything commutes in Fig 4.2 (i.e. you can follow the arrows in any order), cycles in $C_r(K)$ are transformed into cycles in $C_r(L)$, same for boundaries, and therefore $f_\#$ induces an application f_* between $H_r(K)$ and $H_r(L)$ which is linear (since $f_\#$ is). $\square$

Proposition 3 (Functoriality)**.** *We have the following properties:*

(i) Given $K \xrightarrow{f} L \xrightarrow{g} M$, $(g \circ f)_* = g_* \circ f_*$.

(ii) Given K, $(id_K)_* = id_{H_r(K)}$

Proof. Exercise (use the construction of $f_\#$ and the fact that things are preserved by turning it into f_*). $\square$

4.3 Application: invariance of homology

Question: How to go from simplicial complexes to triangulable space? At this point, we only defined homology for a simplicial complex K which is assumed to be a triangulation of some space X. But does the choice of K matter? If it does not, we would like to define $H_r(X) := H_r(K)$ for any choice of triangulation K. Good news: this is actually true!

Proposition 4. *Given X triangulable, for any two triangulations K, L of X, one has $H_r(K) \simeq H_r(L)$ (reminder: $\simeq$ means* is isomorphic to *for two vector spaces).*

Hence, $H_r(X)$ is well-defined.

Proof. Admitted (not too complicated but technical). □

Proposition 5. *If $f, g : X \to Y$ are homotopy equivalent, then $f_* = g_* : H_r(X) \to H_r(Y)$.*

Proof. Admitted (same as before, technical). □

Corollary 2. *If X, Y are two homotopy equivalent spaces, then $H_r(X) \simeq H_r(Y)$. That is:* Homology is an homotopy invariant.

Proof. By definition of being homotopy equivalent for two spaces X, Y, we know that there are two functions $f : X \to Y$ and $g : Y \to X$ s.t. $g \circ f \sim id_X, f \circ g \sim id_Y$ (reminder: $f \sim g$ means there is $\varphi : [0,1] \times X \to Y$ continuous such that $\varphi(0, \cdot) = f(\cdot), \varphi(1, \cdot) = g(\cdot)$).

Due to functoriality and previous proposition:

$$f_* \circ g_* = (f \circ g)_* = (id_X)_* = id_{H_r(X)}$$
$$g_* \circ f_* = (g \circ f)_* = (id_Y)_* = id_{H_r(Y)}$$

That is: $f_* : H_r(X) \to H_r(Y)$ is an isomorphism of vector spaces ($f_*^{-1} = g_*$) and therefore,

$$H_r(X) \simeq H_r(Y)$$

□

Remark: We proved that homology is an homotopy equivalence invariant. The converse is not true: one can find two topological spaces X, Y with $H_r(X, \mathbb{K}) = H_r(Y, \mathbb{K})$ for all r and any field $\mathbb{K}$, but with X, Y being not homotopy equivalent. The *Poincaré homology sphere* is an example: it has the same homology as $\mathcal{S}^3$ but is not homotopy equivalent to it.

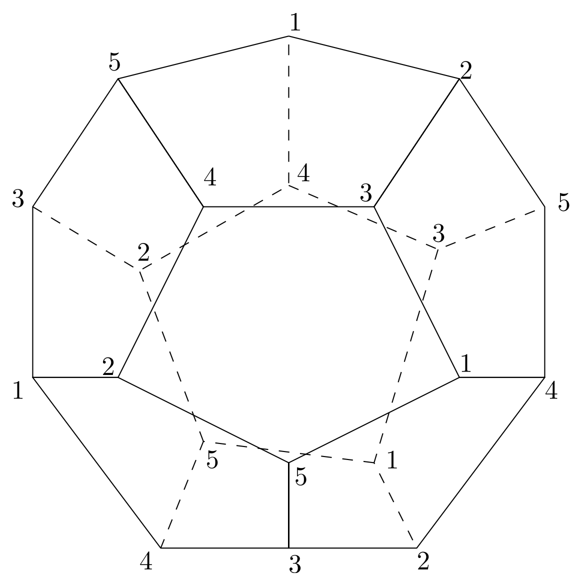

Figure 4.3: Representation of the Poincaré sphere.

4.4 Singular homology (optional)

The goal of this section is to extend the construction of homology we made on triangulable spaces (by defining the homology on simplicial complexes) to a more general class of spaces. The construction is roughly the same as before.

4.4.1 Singular chains

Idea: Build cycles by gluing basic buildings blocs (the *singular simplices*) together into *chains*.

Definition 2. *The k-dimensional standard simplex Δ_k is defined by the convex hull of $\{v_0 \dots v_k\}$ in $\mathbb{R}^k$, where $v_0 \dots v_k$ are linearly independent (that is, $(v_1 - v_0, \dots, v_k - v_0)$ is a basis of $\mathbb{R}^k$).*

Definition 3. *A k-dimensional singular simplex (or* singular k-simplex*) in X is a continuous map $\sigma : \Delta_k \to X$.*

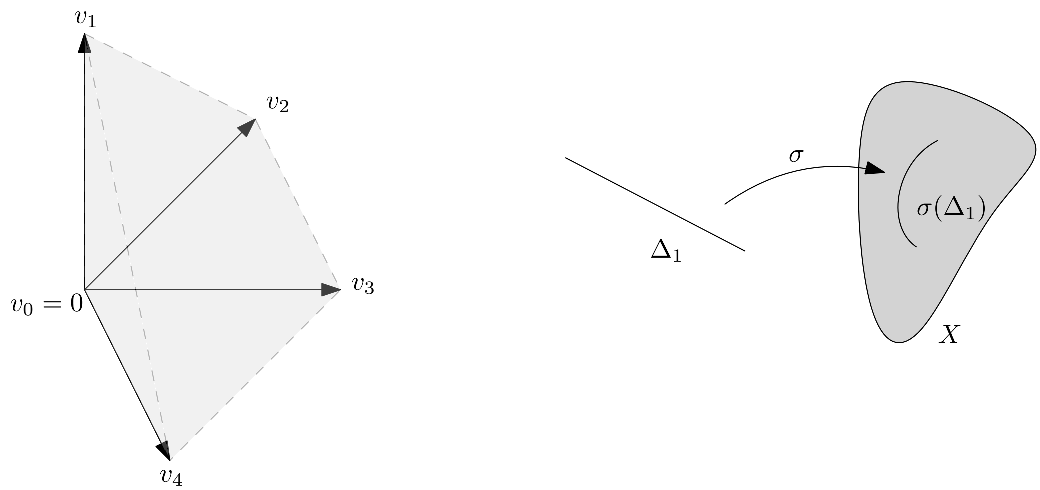

Figure 4.4: *(left)* The k-dimensional standard simplex Δ_k. *(right)* Example of continuous mapping from Δ_1 to some space X.

Let $\mathbb{K}$ be a fixed field.

Definition 4. *A k-chain is a formal finite $\mathbb{K}$-weighted sum of singular k-simplices:*

$$c := \sum_{i=1}^{n} \underbrace{\alpha_i}_{\in \mathbb{K}} \underbrace{\sigma_i}_{:\Delta_k \to X}$$

Equivalent definition: c is a map from the singular k-simplices to $\mathbb{K}$ that is zero except except on finitely many simplices. Here, c is defined by the values $c(\sigma_i) := \alpha_i$, which must be non-zero for only a finite set of $(\sigma_i)_i$.

Note: We often identify a simplex with its image. For example, we will write σ to denote $\sigma(\Delta_k)$.

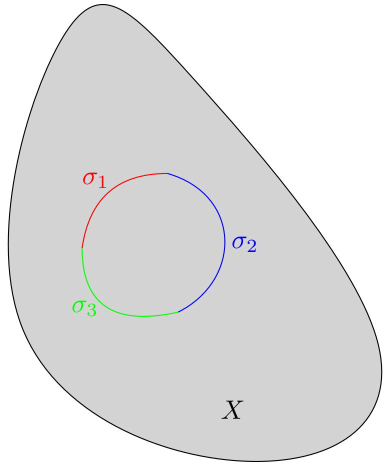

Figure 4.5: $c = \sigma_1 + \sigma_2 + \sigma_3$ is a (1-dimensional) singular chain.

Note: The set of k-chains is the set of $\mathbb{K}$ valued functions over $\mathcal{C}(\Delta_k, X)$ with finite support. It has a vector space structure ($\alpha c + c' : \sigma \mapsto \alpha c(\sigma) + c'(\sigma)$) We call $C_k(X) = \mathbb{K}^{\mathcal{C}(\Delta_k, X)}$ this (huge) vector space.

4.4.2 Boundary operator

Question: What is the boundary of a chain? The idea is:

- Define the boundary of a simplex
- extend to chains by linearity

As fpr simplicial homology, we can define the orientation of simplices.

Definition 5 (Orientation)**.** *An* orientation *of Δ_k is an order $[v_{\pi(0)} \dots v_{\pi(k)}]$ on the vertices of Δ_k, where $\pi \in \mathfrak{S}_{k+1}$ is a permutation.*

Two orientations π, π' are equivalent *if $\pi' \circ \pi$ has positive signature.*

There are two equivalence classes of orientation:

- *Those with positive signature (class of id)*
- *Those with negative signature.*

Definition 6 (Boundary operator)**.**

$$\partial \Delta_k = \sum_{j=0}^{k} (-1)^j [v_0 \dots, \widehat{v_j}, \dots, v_k]$$

where $(-1)[v_0 \dots v_k]$ denote the simplex convex hull$(\{v_0 \dots v_k\})$ with the opposite orientation.

Then, given $\sigma : \Delta_k \to X$,

$$\partial \sigma := \sum_{j=0}^{k} (-1)^j \sigma\big|_{[v_0 \dots, \widehat{v_j}, \dots v_k]}$$

Given $c = \sum \alpha_i \sigma_i \in C_k(X)$,

$$\partial c := \sum \alpha_i \partial \sigma_i \in C_{k-1}(X)$$

4.4.3 Singular homology groups

Just as in previous lecture (section simplicial homology), we can define the (singular) homology group $H_r^{\text{sing}}(X)$ for a space X and all equivalent notions (Betti numbers, etc.). We have the following theorem:

Theorem 1 (Equivalence of homology)**.** *For any triangulable space X, $H_r(X) = H_r^{\text{sing}}(X)$*

So the take home message is that the notion of homology extend to a class of spaces larger than the simplicial complexes only.